

Robustness of supersensitivity to small signals in nonlinear dynamical systems

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Nonlinear dynamical systems possessing an invariant subspace can display interesting dynamical behavior, such as on-off intermittency and bubbling. This Rapid Communication shows that a class of such systems has amazing features of (1) supersensitivity to small input signals and (2) robustness of the supersensitivity in the presence of noise. These features make the systems very promising as small signal detectors.
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Nonlinear dynamical systems with an invariant subspace due to symmetries or other constraints are of great interest. An example is the synchronization of coupled chaotic systems [1]. Such systems can display interesting and unusual dynamical behaviors, such as on-off intermittency [2] and bubbling [3]. In on-off intermittency, the invariant manifold is slightly unstable and the system can remain close to the invariant manifold for long periods of time, interrupted only by some occasional large bursts away from the invariant manifold. In bubbling, the invariant manifold is stable. However, there are unstable invariant sets embedded in the chaotic sets, and small perturbations of noise or parameter mismatches can result in large intermittent bursts. This can be harmful in connection with applications of synchronization, such as in secure communication [4], because high quality synchronization is destroyed by bubbling [3].

The purpose of this Rapid Communication is to show that such systems are very sensitive to small constant or time-dependent input signals. With an additional symmetrical condition, the sensitivity is robust to external noise, which makes the systems very promising for potential applications to weak signal detection.

In spite of the variety of such systems, their behavior can often be described by the following equations:

$$y_{n+1} = G(x_n, y_n, a), \quad (1)$$

$$x_{n+1} = F(x_n, y_n, a), \quad (2)$$

where $G(x_n, 0, a) = 0$ and the variables y_n and x_n represent the distance from the invariant manifold $y = 0$ and the dynamics within the invariant subspace, respectively. In general, y and x are vectors. Note that many generic properties of the above phenomena can be observed in some very simple systems, and we consider x and y as one-dimensional variables. Here a is a parameter that may change the dynamics within the invariant subspace as well as their stability. We are interested in the behavior of the system close to the invariant manifold where the systems can be represented by the approximate linear dynamics:

$$y_{n+1} = g(x_n, a)y_n + O(y_n^2), \quad (3)$$

$$x_{n+1} = f(x_n, a) + O(y_n). \quad (4)$$

The nonlinearity of the systems serves to keep the solution bounded. Usually, chaotic signals have quickly (exponentially) decaying correlation. It is often plausible to assume that the chaotic signals are not correlated when considering the long time behavior of the system in Eq. (3). Based on this assumption, and for the sake of simplicity and without loss of generality, we are led to consider the behavior of the following simple random driven map:

$$y_{n+1} = ax_n y_n + O(y_n^2), \quad (5)$$

where x_n is a random driving signal. Introducing the variable $z_n = \ln|y_n|$, Eq. (5) becomes

$$z_{n+1} = z_n + \ln|x_n| + \ln|a|. \quad (6)$$

The critical value of the parameter a is defined by $\ln|a_c| + \langle \ln|x_n| \rangle = 0$, where $\langle \dots \rangle$ represents the time average. Close to the critical point $\delta = (a - a_c)/a_c \ll 1$, Eq. (6) can be rewritten as

$$z_{n+1} = z_n + \delta + \xi_n, \quad (7)$$

where $\xi_n = \ln|x_n| - \langle \ln|x_n| \rangle$ is a random variable with vanishing mean and variance D .

To analyze the long time behavior, map (7) can be replaced by the corresponding stochastic differential equation, which is the equation for Brownian motion in one dimension with a drift. The corresponding Fokker-Planck equation is

$$\frac{\partial W}{\partial t} = -\delta \frac{\partial W}{\partial z} + \frac{D}{2} \frac{\partial^2 W}{\partial z^2}, \quad (8)$$

and the static solution of the probability density W is $W(z) = C \exp(\alpha z)$ with $\alpha = 2\delta/D$. In the variable y , the distribution becomes $W(y) = C|y|^{\alpha-1}$.

Now let us consider that there is a small positive constant input p to the system of Eq. (5), namely, $y_{n+1} = ax_n y_n + O(y_n^2) + p$, where p is the order of 10^{-m} , $m \gg 1$ (suppose the maximal value of y_n has the order of unity). For simplicity, we suppose that $y_n > 0$ for initial value $y_0 > 0$ and $p > 0$. For $y_n \gg p$, the behavior of the system is governed by Eqs. (5) and (7). The effect of the small input can be regarded as a reflecting barrier to the Brownian motion of the system (7), i.e., $z \geq -m$. On the other hand, the state of the

system is bounded by the nonlinearity of the system. We can introduce a parameter τ to represent the effect of up-boundary of the system. Based on these considerations, the behavior of the system with input p can be understood by the Brownian motion confined between the two boundaries. The property of the system is determined by the competition between the constant drift δ and the diffusion D . The diffusion is dominant for $\delta \sim 0$, i.e., for parameter a close to the critical point a_c where the system displays on-off intermittency (bubbling) and can access to both the lower and upper boundaries frequently, becoming sensitive to the small input and producing large bursts. Otherwise, the drift becomes dominant when a is far away from the critical point; for $a < a_c$, the system will spend most of its time close to the lower boundary and produce rare large bursts, and the small input does not lead to significant large output in the system; for $a > a_c$, the system will spend most of its time close to the upper boundary, and will rarely access the level of the input; also, the small input does not have significant effects on the system behavior. We can expect that the system is sensitive to small input when it is on-off intermittent.

The above consideration leads to the normalization condition $\int_p^\tau C y^{\alpha-1} dy = 1$, which gives $C = \alpha / (\tau^\alpha - p^\alpha)$. Now we can estimate the amplitude of the output signals by the ensemble average

$$\langle y \rangle = C \int_p^\tau y^\alpha dy = \frac{\alpha}{1 + \alpha} \frac{\tau^{\beta-p}}{\beta - 1}, \quad (9)$$

where $\beta = (\tau/p)^\alpha$.

If $\beta \approx 1$, the small input can change the behavior of the system greatly. For the conditions $|\alpha| \ll 1$, $\tau \gg p$ and $|\alpha| \ln(\tau/p) \ll 1$, one has $\beta \approx 1 + \alpha \ln(\tau/p)$ and

$$\langle y \rangle \approx \frac{\tau}{\ln(\tau/p)}. \quad (10)$$

Equation (10) shows that the average value decreases to zero with the decrease of input p only logarithmically, suggesting that close to the critical point, a very small input p can produce a relatively large output, i.e., the system is supersensitive to small input. A measure of the sensitivity can be

$$S = \frac{\langle y \rangle}{p} = \frac{\tau}{p \ln(\tau/p)}. \quad (11)$$

For example, with $\tau = 1$ and $p = 10^{-15}$, the value of S is about 2.9×10^{13} .

To demonstrate the validity of the above analysis, we carry out simulations with the following two systems of the form $y_{n+1} = ax_n f(y) + p$. For system I, $f(y)$ is a piecewise linear map

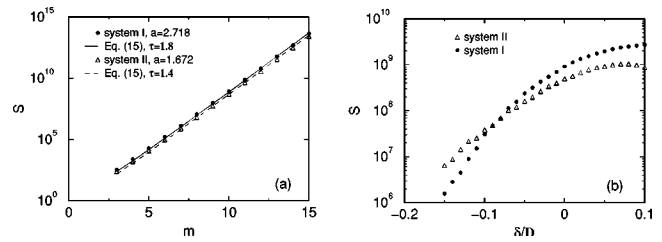


FIG. 1. (a) Dependence of the sensitivity S close to the critical point of the systems on input $p = 10^{-m}$. (b) Dependence of S on δ for $p = 10^{-10}$.

$$f(y) = \begin{cases} \frac{c_1}{c_2}(-c_1 - c_2 - y), & y < -c_1 \\ y, & |y| \leq c_1 \\ \frac{c_1}{c_2}(c_1 + c_2 - y), & y > c_1, \end{cases} \quad (12)$$

where the parameters c_1 and c_2 are chosen so that $y_n > 0$ for the positive initial value y_0 and p , i.e., the bursting in the system is symmetry breaking [5]. We use $c_1 = 1$ and $c_2 = 2$ in our simulations, with x_n uniform on $[0, 1]$ and thus $a_c = e = 2.71828 \dots$ and $D = 1$. For system II, $f(y) = \sin(y)$, and x_n is a chaotic signal generated by the logistic map $x_{n+1} = 3.75x_n(1 - x_n)$, which gives x_n a distribution with singularities, and $a_c \approx 1.673$ and $D = 0.2$. S is estimated close to the critical point for these two systems. With $p = 10^{-m}$, S as a function of m is shown in Fig. 1(a). The analytical estimation in Eq. (11) with $\tau = 1.8$ and $\tau = 1.4$ gives good approximation to the simulation results. Figure 1(b) shows the dependence of S on parameter deviation δ from the critical point. The sensitivity is maintained over a large range of the parameter a .

The feature of supersensitivity is maintained even for time-dependent signals, for example,

$$p_{n+N} = p_n = \begin{cases} p, & 0 < n \leq N/2 \\ -p, & N/2 < n \leq N. \end{cases} \quad (13)$$

In order for the system to have a symmetrical response property to positive and negative inputs, we require that the map $f(y)$ have odd symmetry $f(-y) = -f(y)$, and on-off intermittency in the system is symmetry breaking, so that a positive (negative) small input will eventually lead to only positive (negative) output. For $N \gg N_0$, where N_0 is the relaxation time of the system, the sensitivity can be measured by Eq. (11).

The feature of supersensitivity makes the systems very promising for application as a sensitive device for small signals. In the context of application, we should consider the behavior of the system in the presence of additive noise, namely, $y_{n+1} = ax_n f(y_n) + p_n + e_n$, where e_n is a small Gaussian white noise with zero mean and standard deviation σ . We can study the long time behavior of the system by the corresponding stochastic differential equation $dy/dt = (\delta + \xi)y + p + e$. The Fokker-Planck equation is

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial y} \left[\left(\delta + \frac{D}{2} \right) y W \right] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [(Dy^2 + \sigma^2) W], \quad (14)$$

and the static solution under the adiabatic condition $N \gg N_0$ is given by

$$W(y) = C \left(y^2 + \frac{\sigma^2}{D} \right)^{(\alpha-1)/2} \exp \left[\frac{2p}{\sqrt{D}\sigma} \arctan \frac{\sqrt{D}y}{\sigma} \right]. \quad (15)$$

This distribution, however, is very complicated for evaluating $\langle y \rangle$. To simplify the calculation, we employ the similar heuristic boundary conditions in the above. Under the conditions $\sigma \ll \sqrt{D}$, $p \sim \sigma$, we approximate the distribution by

$$W(y) = \begin{cases} C|y|^{\alpha-1} \exp \left[\frac{\pi p}{\sqrt{D}\sigma} \operatorname{sgn}(y) \right], & |y| \geq p \\ 0, & |y| < p. \end{cases} \quad (16)$$

It is clear that, by the limit $\sigma \rightarrow 0$, we come back to the result for the noise-free case. With this approximation, we obtain that, close to the critical point,

$$\langle y \rangle \approx \frac{\int_{-\tau}^{\tau} y W(y) dy}{\int_{-\tau}^{\tau} W(y) dy} = \frac{\tau}{\ln(\tau/p)} \tanh \left(\frac{\pi}{\sqrt{D}} R \right), \quad (17)$$

where $R = p/\sigma$ provides a natural measure of the signal-to-noise ratio.

The above analysis is demonstrated by numerical simulations in the presence of noise. Figure 2(a) is a typical response of the system to a noisy small signal, and Fig. 2(b) shows $\langle y_n \rangle$ for different values of R . Figure 2(c) displays the dependence of $\langle y \rangle$ on R . It is apparent that the above approximate analysis gives a good account of the results in a large range of R . Over a wide range of R , $\langle y \rangle$ is very close to that of the noise-free case. The supersensitivity is thus robust to additive noise. This feature of sensitivity is quite different from that of the sensitivity near the onset of a period-doubling bifurcation in many dynamical systems [6]. There the system is only sensitive to perturbations near half the fundamental frequency of the system for a bifurcation parameter very close to the onset point.

To examine the performance of the system as a small signal detector, we calculate the probability of bit error P_b in the presence of additive noise. The detection is done by examining the time average of the output y_n in the duration of an input bit b_k , namely $s_k = 1/N \sum_{N(k-1)+1}^{Nk} y_n$. A bit b_k is detected as $B_k = 1$ (-1) if $s_k > 0$ (< 0). For $N \gg N_0$, the variable s_k is expected to fluctuate around $\langle y \rangle$. Although y_n cannot be assumed to be uncorrelated, for very large N , it might still be plausible to assume that s_k approaches a Gaussian distribution with an average $\langle y \rangle$ and a variance $D_N = \Delta/N$, especially in the case in which R is small and y_n has a comparable distribution to positive and negative values. Based on this assumption, P_b can be evaluated approximately as

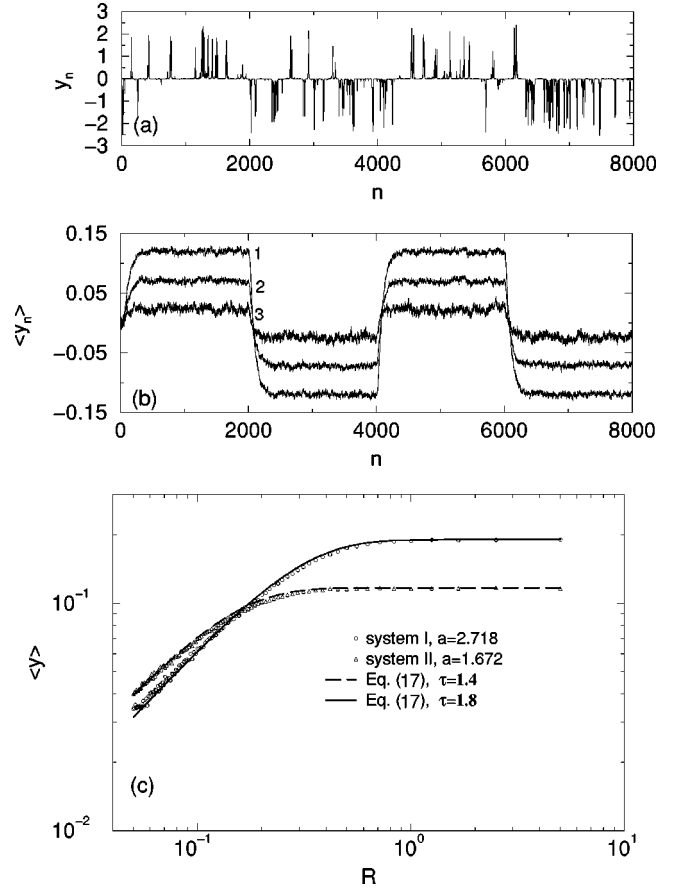


FIG. 2. (a) An example of the bursting behavior of the system I with $a=2.6$, $N=4000$, $p=10^{-5}$, and $R=0.1$. (b) Time series of the ensemble average $\langle y_n \rangle$ over 5000 samples for the system I with $a=2.6$, $p=10^{-5}$, and $N=4000$. The three plots are (1) for the noise-free case, (2) for $R=0.2$, and (3) for $R=0.05$. (c) Ensemble average $\langle y \rangle$ close to the critical point as a function of R for constant input $p=10^{-5}$. The solid lines are an estimation of Eq. (17).

$$P_b = \text{Prob}(B_k \neq b_k) = \text{Prob}(s_k b_k < 0)$$

$$\approx \frac{1}{2} \left[1 - \operatorname{erf} \left(\sqrt{\frac{N}{\Delta}} \langle y \rangle \right) \right], \quad (18)$$

where $\langle y \rangle$ is given by Eq. (17).

In our simulations, we estimate P_b with 10^6 random bits in system I at $a=2.6$ for input levels $p=10^{-4}$ and $p=10^{-6}$. The quantities N_0 and Δ are estimated in simulation with constant input, giving $N_0=350$, $\Delta=5$ for $p=10^{-4}$ and $N_0=700$, and $\Delta=4$ for $p=10^{-6}$. Both the results of P_b from simulations and from estimation in Eq. (18) are shown in Fig. 3 for $N=3N_0, 5N_0, 10N_0, 15N_0$. The parameter τ , used to fit Eq. (18) to the simulation results, is $\tau=1.4$ for $p=10^{-4}$ and $\tau=1.1$ for $p=10^{-6}$. It is apparent that the estimation can be quite good for N much larger than N_0 . For N comparable to N_0 , the effects of the transient process during the relaxation time becomes significant, and the estimation deviates from the simulation results. For R getting larger, the bursting behavior becomes more asymmetrical, s_k can no longer be approximated by a Gaussian distribution, and the estimation also deviates from the simulations results.

The above results show that detection error can be quite low even for a small signal with a level much lower than the

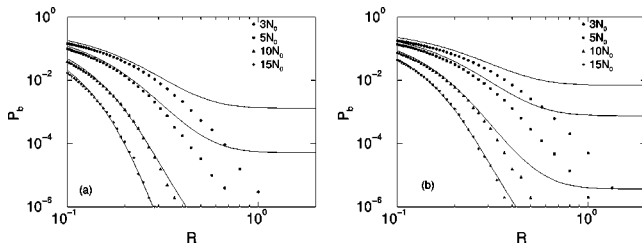


FIG. 3. Probability of bit error P_b as a function of R for different levels of input and different bit durations. The solid lines are an estimation of Eq. (18). (a) $p=10^{-4}$, $N_0=350$, $\Delta=5$, and $\tau=1.4$. (b) $p=10^{-6}$, $N_0=700$, $\Delta=4$, and $\tau=1.1$.

environment noise if the signal has a bit duration much larger than the relaxation time of the system. Related to this, there seems to be a frequency cutoff above which detection becomes unreliable. This frequency gets larger as the input level increases, because the relaxation time becomes shorter for a higher level of input.

The above properties of supersensitivity and its robustness in the presence of noise is universal in a general class of coupled symmetrical systems displaying on-off intermittency with symmetry breaking. The sensitivity is due to the power-law distribution of on-off intermittency of y in a wide interval $10^{-m} < |y| < \tau$, so that the system can have access to the level of the small input, and at the same time produce frequent large bursts. If the maps are not coupled to random or chaotic driving x_n , there is no diffusion in the system (D

$=0$), and the small input does not have a significant effect on the system output. For uncoupled non-on-off maps, namely, $y_{n+1} = af(y_n)$, if $a < 1$ (for the maps in the above), the fixed point $y=0$ is stable, and the small input cannot produce a large output at all; if $a > 1$, the state y_n can no longer come to the level of a small input of the order $p=10^{-m}$ ($m > 1$) with significant frequency, and the output will not manifest the small input. In both cases, the systems do not possess the sensitivity in the coupled, on-off intermittent systems. The symmetry breaking of the bursting also plays an important role in the sensitivity and robustness because, under this condition, a transition of the state between $y > 0$ and $y < 0$ is determined only by the switch of the small signal. If the bursting is not symmetry breaking, there are additional transitions between $y > 0$ and $y < 0$ induced by bursting states, which will degrade the sensitivity and robustness.

In conclusion, we demonstrate that a class of nonlinear dynamical systems, having an invariant subspace and displaying on-off intermittency and bubbling, have the feature of supersensitivity to small constant or time-dependent input signals. With an additional odd symmetry condition, the sensitivity is robust to additive noise. The features make the systems very promising for such useful applications as sensitive devices.

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